ON THE VARIATIONAL METHODS OF SOLUTION OF PROBLEMS OF STEADY-STATE CREEP IN PLATES AND SHELLS IN THE CASE OF FINITE DISPLACEMENTS

(K VARIATSIONNYM METODAM RESHENIIA ZADACH USTANOVIVSHEISIA POLZUCHESTI PLASTIN I OBOLOCHEK V SLUCHAE KONECHNYKH PEREMESHCHENII)

PMM Vol. 26, No. 3, 1962, pp. 492-496

I. G. TEREGULOV
(Kazan')

(Received December 11, 1961)

The variational principle of virtual velocities (principle of stationary total rate of work) is introduced, under the assumption of steady-state creep, with finite displacements and small extensions, as compared to unity. This principle represents a generalization of the corresponding statement proved for the geometrically linear case in [1]. As an illustration of the use of the formulated principle, the problem of bending of a thin circular plate in the conditions of steady-state creep is considered.

In the derivations no distinction is made between the deformed and the undeformed states in taking volume and surface integrals of functions which do not depend upon the orientation of the respective volumes and surfaces. This introduces an error whose order of magnitude is not larger than that of the strain compared to unity. Likewise, in performing the differentiation we will make no distinction between the metric tensors of the deformed and the undeformed states.

1. Let us designate by δN the rate of work of the external surface loads $\mathbf{P}$ and body forces $\mathbf{Q}$ on the variations of velocities $\delta\mathbf{v}$ admissible by the geometric constraints

$$\delta N = \iint_S \mathbf{P}\cdot\delta\mathbf{v}\,dS + \iiint_V \mathbf{Q}\cdot\delta\mathbf{v}\,dV \tag{1.1}$$

Here, S is the boundary of the volume V occupied by the body. In this volume let us introduce a parametrization x^i $(i = 1,\ 2,\ 3)$ with the base vectors $\mathbf{r}_i = \partial\mathbf{r}/\partial x^i$ and with the metric tensor $g_{ik} = \mathbf{r}_i \times \mathbf{r}_k$, where $\mathbf{r}$ is the position vector of the points in the body before deformation.

Let $\mathbf{r}^*$ be the position vector of the points in the body after deformation. Then $\mathbf{r}^* = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ is the displacement vector. Note that the Latin indices of tensor character take the values 1, 2, 3, whereas the Greek ones - the values 1 and 2.

Let us designate the rate of work of the internal stresses σ^{ik} in the variations of the creep strain-rates $\delta\xi_{ik}$ by δM

$$\delta M = \iiint\limits_V \sigma^{ik}\,\delta\xi_{ik}\,dV \tag{1.2}$$

The following statement is valid: of all the velocities in the body which are admissible by the geometric constraints, the ones that actually occur are those which satisfy the condition

$$\delta J = 0, \qquad J = M - N \tag{1.3}$$

Since for the strains we have

$$2\varepsilon_{ik} = \frac{\partial \mathbf{u}}{\partial x^i}\cdot \mathbf{r}_k + \frac{\partial \mathbf{u}}{\partial x^k}\cdot \mathbf{r}_i + \frac{\partial \mathbf{u}}{\partial x^i}\cdot\frac{\partial \mathbf{u}}{\partial x^k} \tag{1.4}$$

for the strain-rates we obtain

$$2\xi_{ik} = 2\frac{\partial \varepsilon_{ik}}{\partial t} = \frac{\partial \mathbf{v}}{\partial x^i}\cdot \mathbf{r}_k + \frac{\partial \mathbf{v}}{\partial x^k}\cdot \mathbf{r}_i + \frac{\partial \mathbf{v}}{\partial x^i}\cdot\frac{\partial \mathbf{u}}{\partial x^k} + \frac{\partial \mathbf{u}}{\partial x^i}\cdot\frac{\partial \mathbf{v}}{\partial x^k} \tag{1.5}$$

where $\mathbf{v} = \partial\mathbf{u}/\partial t$ is the velocity vector of the points in the body, t is the time. By our assumption the geometric constraints are satisfied and the relations (1.4) and (1.5) are valid. We will show that, based on these relations, the equations of equilibrium (motion) and the natural static boundary conditions follow from the variational equation (1.3), and, inversely, by satisfying the equations of equilibrium and static boundary conditions, and maintaining the geometric constraints, we obtain the Equation (1.3). The validity of our original statement will follow from here.

Substituting $\delta\xi_{ik}$ according to (1.5) into the variational equation (1.3), bearing in mind (1.1) and (1.2) and (since only the variations of velocities are allowed)

$$2\delta\xi_{ik} = \left(\mathbf{r}_i + \frac{\partial \mathbf{u}}{\partial x^i}\right)\cdot\delta\frac{\partial \mathbf{v}}{\partial x^k} + \left(\mathbf{r}_k + \frac{\partial \mathbf{u}}{\partial x^k}\right)\cdot\delta\frac{\partial \mathbf{v}}{\partial x^i} = \mathbf{r}_i^*\cdot\delta\mathbf{v}_k + \mathbf{r}_k^*\cdot\delta\mathbf{v}_i \tag{1.6}$$

we obtain

$$\delta J=\iiint\limits_V \sigma^{ik}\mathbf{r}_k^*\cdot\delta\mathbf{v}_i dV-\iiint\limits_V \mathbf{Q}\cdot\delta\mathbf{v}dV-\iint\limits_S \mathbf{P}\cdot\delta\mathbf{v}dS \tag{1.7}$$

Here $\mathbf{v}_i=\partial\mathbf{v}/\partial x^i$, σ^{ik} are contravariant components of the stress tensor. Applying the formula of Ostrogradsky-Gauss to the first term on the right-hand side of the relation (1.7) we obtain

$$\iiint\limits_V \sigma^{ik}\mathbf{r}^*\cdot\delta\mathbf{v}_i dV=-\iiint\limits_V (\nabla_i\sigma^{ik}\mathbf{r}_k^*)\cdot\delta\mathbf{v}\,dV-\iint\limits_S \sigma^{ik}n_i\mathbf{r}_k^*\cdot\delta\mathbf{v}\,dS \tag{1.8}$$

where $\nabla_i(\ldots)$ is the symbol of the covariant derivative with respect to the metric tensor g_{ik}, n_i are covariant components of the interior normal unit vector on the surface S. δJ now takes the form

$$\delta J=-\iiint\limits_V \{\nabla_i(\sigma^{ik}\mathbf{r}_k^*)+\mathbf{Q}\}\cdot\delta\mathbf{v}\,dV-\iint\limits_S (\sigma^{ik}\mathbf{r}_k^*n_i+\mathbf{P})\cdot\delta\mathbf{v}dS \tag{1.9}$$

It follows from the last relation that, if all the static conditions are satisfied and no geometric and kinematic constraints are violated, we have $\delta J=0$. And vice versa, the equations of equilibrium (motion) and natural static boundary conditions follow from $\delta J=0$.

With additional limitations imposed on the state of stress and deformation, naturally, a more powerful result has been obtained in [2].

If the exponential law is used for the steady-state creep [1], the variational equation (1.3) will have the form

$$\delta\iiint\limits_V \frac{H^{\mu+1}}{(1+\mu)B^{\mu}}dV-\delta\iiint\limits_V \mathbf{Q}\cdot\mathbf{v}\,dV-\delta\iint\limits_S \mathbf{P}\cdot\mathbf{v}dS=0 \tag{1.10}$$

Here B is a function of time and temperature, which is determined experimentally, μ is a constant, H is the intensity of the shear strain-rates, which is expressed in the form

$$H^2=2\xi_{ik}\xi^{ik} \qquad \text{(it is assumed that } \xi_{ik}g^{ik}=0) \tag{1.11}$$

and the relation between the stresses and the strain rates is given by the expressions

$$\sigma^{ik}-\sigma g^{ik}=\frac{1}{(1+\mu)B^{\mu}}\frac{\partial H^{\mu+1}}{\partial\xi_{ik}}, \qquad 3\sigma=\sigma^{ik}g_{ik} \tag{1.12}$$

It can be shown that $\delta^2 J\geqslant 0$ if the law (1.12) is assumed (see [1, p.109]).

2. In forming the variational equation for thin plates and shells

(single-layered) we are going to start with the usual assumptions that the normal components of stress on the elemental areas parallel to the middle surface S_0 are small, as compared to the normal components of stress on the elemental areas perpendicular to the middle surface, and that the shear strains ε_{13} and ε_{23} are absent. Here $x^3 = z$ is the coordinate measured along the normal to the middle surface, x^1, x^2 are curvilinear coordinates on the surface S_0.

The displacement vector of the points in the shell has the following form

$$\mathbf{u} = (u_\alpha - z\nabla_\alpha w)\boldsymbol{\rho}^\alpha + w\mathbf{m} \tag{2.1}$$

where $\boldsymbol{\rho}_\alpha$ are base vectors on the surface S_0 in the parametrization x^α ($\alpha = 1, 2$), $\mathbf{m}$ is the unit vector, normal to the surface S_0, which is determined from the relation

$$\mathbf{m}c_{\alpha\beta} = \boldsymbol{\rho}_\alpha \times \boldsymbol{\rho}_\beta \tag{2.2}$$

$$c_{12} = -c_{21} = \sqrt{a}, \qquad c_{11} = c_{22} = 0, \ a = \det(a_{\alpha\beta}), \ a_{\alpha\beta} = \boldsymbol{\rho}_\alpha \cdot \boldsymbol{\rho}_\beta$$

The meaning of the quantities u_α and w is clear from the form in which the displacement vector is written down, (2.1). On the basis of the Formulas (1.5) we obtain

$$\xi_{\alpha\beta} \approx \xi_{\alpha\beta}^0 - z(\nabla_\alpha\dot{\omega}_\beta + \nabla_\beta\dot{\omega}_\alpha), \qquad 2\xi_{\alpha\beta}^0 \approx \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - 2b_{\alpha\beta}v + \dot{\omega}_\alpha\omega_\beta + \omega_\alpha\dot{\omega}_\beta \tag{2.3}$$

$$\dot{\omega}_\alpha = \frac{\partial\omega_\alpha}{\partial t}, \quad v^\beta = \frac{\partial u_\beta}{\partial t}, \quad v = \frac{\partial w}{\partial t}, \quad b_{\alpha\beta} = -\boldsymbol{\rho}_\alpha \cdot \frac{\partial \mathbf{m}}{\partial x^\beta}, \quad \omega_\alpha = \nabla_\alpha w + b_\alpha^\beta v_\beta$$

Here the expressions for $\xi_{\alpha\beta}$ are simplified, with an error of the order of magnitude not larger than that of the strain, and not larger than h/R, as compared to unity, where $2h$ is the thickness of the plate or shell, R is the smaller of the radii of curvature of the middle surface, $\nabla_\alpha(\ldots)$ is the symbol of the covariant derivative with respect to the metric tensor $a_{\alpha\beta}$.

On the basis of the assumption $\sigma^{33} \ll \sigma^{\alpha\alpha}\sqrt{(a_{\alpha\alpha})}\sqrt{(a_{\alpha\alpha})}$ from physical relations (1.12) we obtain

$$\sigma^{\alpha\beta} - \sigma_* a^{\alpha\beta} = \frac{2}{B^\mu} H^{\mu-1}\xi^{\alpha\beta}, \quad 3\sigma_* = \sigma^{\alpha\beta}a_{\alpha\beta}, \quad \sigma^{\alpha\beta} = \frac{2}{B^\mu} H^{\mu-1}(\xi^{\alpha\beta} + a^{\beta\alpha}\xi_{\varepsilon\gamma}a^{\varepsilon\gamma}) \tag{2.4}$$

the relation

$$\sigma^{33} - g^{33}\sigma = \frac{2}{B^\mu} H^{\mu-1}\xi_{33}$$

is identically satisfied within the limits of the accepted accuracy due to the condition $\xi_{ik}g^{ik} = 0$. The last condition expresses the incompressibility of the material in the state of creep and is highly accurate.

On the basis of the condition $\xi_{ik}g^{ik} = 0$, for the intensity of the shear strain-rates, remembering that $\xi_{13} = \xi_{23} = 0$, we obtain

$$H_*^2 = 2(\xi_{\alpha\beta}\xi^{\alpha\beta} + a_{\rho\gamma}\xi^{\rho\gamma}a_{\alpha\beta}\xi^{\alpha\beta}) \qquad \sigma^{\alpha\beta} = \frac{1}{(1+\mu)B^\mu}\frac{\partial H_*^{\mu+1}}{\partial \xi_{\alpha\beta}} \tag{2.5}$$

Thus, for thin shells we obtain the variational equation of the principle of stationary total rate of work for steady-state creep in the form

$$\delta\iint_{S_0}\int_{-h}^{h}\frac{H_*^{\mu+1}}{(1+\mu)B^\mu}dzdS_0 - \delta\iint_{S_+}\mathbf{P}_+\{(v_\alpha - h\nabla_\alpha v)\boldsymbol{\rho}^\alpha + v\mathbf{m}\}dS \tag{2.6}$$

$$\delta\iint_{S_-}\mathbf{P}_-\{(v_\alpha + h\nabla_\alpha v)\boldsymbol{\rho}^\alpha + v\mathbf{m}\}dS_0 - \delta\iint_{C}\int_{-h}^{h}\mathbf{P}_c\{(v_\alpha - z\nabla_\alpha v)\boldsymbol{\rho}^\alpha + v\mathbf{m}\}dz\,dC = 0$$

Here $2h$ = const is the thickness of the shell, S_+ and S_- are the surfaces $z = h$ and $z = -h$ respectively, $\mathbf{P}_+$ and $\mathbf{P}_-$ are loads on the surfaces S_+ and S_-; the boundary of the surface S_0 is designated by C, and $\mathbf{P}_c$ denotes the vector of external loads applied to the boundary section of the shell $C(-h \leqslant z \leqslant h)$. The body forces have been neglected.

3. Consider a circular plate of radius r, which is subjected to a transverse load of intensity q, and which deforms symmetrically. In this case

$$a_{11} = r^2, \qquad a_{22} = r^2\eta^2 \qquad (0 \leqslant \eta \leqslant 1)$$

$$\Gamma_{22}^1 = -\eta, \qquad \Gamma_{12}^2 = \frac{1}{\eta}, \quad b_{\alpha\beta} = 0, \quad \xi_{12} = 0 \tag{3.1}$$

$$a^{11}\xi_{11} = \frac{1}{r}\frac{\partial v_1}{\partial \eta} + \frac{1}{r^2}\frac{\partial v}{\partial \eta}\frac{\partial w}{\partial \eta} - z\frac{1}{r^2}\frac{\partial^2 v}{\partial \eta^2}, \qquad a^{22}\xi_{22} = \frac{1}{r}\frac{v_1}{\eta} - z\frac{1}{r^2}\frac{1}{\eta}\frac{\partial v}{\partial \eta}$$

Here $\Gamma_{\alpha\beta}^{\omega}$ are the Christoffel symbols with the metric tensor $a_{\alpha\beta}$. We are assuming that the plate is rigidly built-in along its contour (without cut-out portions), so that along the line $\eta = 1$ the following conditions are valid $w = 0$, $u_1 = 0$, $\partial w/\partial\eta = 0$.

Let us choose the functions of displacements, which are the solution of the corresponding geometrically nonlinear problem in the theory of elasticity (see, for instance, [3]), to be the approximating functions for the solution of our problem

$$w = w_0 (1 - \eta^2)^2, \qquad v = \frac{w_0^2}{r}\left(\frac{7}{12}\eta - \frac{5}{2}\eta^3 + 3\eta^5 - \frac{13}{12}\eta^7\right) \tag{3.2}$$

For ξ_{11} and ξ_{22} we obtain from the Formulas (3.1)

$$\begin{aligned} \xi_{11}a^{11} &= \frac{w_0\dot{w}_0}{r^2}\left\{\frac{7}{6} + \eta^2 - 2\eta^4 + \frac{5}{6}\eta^6\right\} + \frac{z\dot{w}_0}{r^2}\, 4(1 - 3\eta^2) \\ \xi_{22}a^{22} &= \frac{w_0\dot{w}_0}{r^2}\left\{\frac{7}{6} - 5\eta^2 + 6\eta^4 - \frac{13}{6}\eta^6\right\} + \frac{z\dot{w}_0}{r^2}\, 4(1 - \eta^2) \end{aligned} \tag{3.3}$$

Here and below, a dot placed above the letter designates a derivative with respect to time. In this case the variational equation (2.6) will have the form

$$\delta \iint\limits_{S_0}\int\limits_{-h}^{h} \frac{B^{-\mu}}{\mu + 1} H_*^{\mu+1}\, dz dS_0 - \delta \iint\limits_{S_0} q\dot{w} dS_0 = 0 \tag{3.4}$$

where

$$H_*^2 = 4\{(a_{11}\xi^{11})^2 + (a_{22}\xi^{22})^2 + a_{11}a_{22}\xi^{11}\xi^{22}\}$$

For the determination of $\dot{w}_0$ we have the equation

$$\frac{\dot{\alpha}}{q_*^m} = \left\{\int\limits_0^1 \eta d\eta \int\limits_{-1}^{1} [\zeta_{11}^2 + \zeta_{22}^2 + \zeta_{11}\zeta_{22}]^{\frac{m+1}{2m}}\right\}^{-m} \tag{3.5}$$

where

$$\begin{aligned} \zeta_{11} &= \alpha\left(\frac{7}{6} + \eta^2 - 2\eta^4 + \frac{5}{6}\eta^6\right) + 2\zeta(1 - 3\eta^2) = \xi_{11}a^{11}\frac{r^2}{2h\dot{w}_0} \\ \zeta_{22} &= \alpha\left(\frac{7}{6} - 5\eta^2 + 6\eta^4 - \frac{13}{6}\eta^6\right) + 2\zeta(1 - \eta^2) = \xi_{22}a^{22}\frac{r^2}{2h\dot{w}_0} \\ q_* &= q_3\frac{B^\mu}{3(8h^2/r^2)^{\mu+1}}, \qquad m = \frac{1}{\mu}, \qquad \zeta = \frac{z}{h}, \qquad \alpha = \frac{w_0}{2h} \end{aligned}$$

From the Formula (3.5) the graphs of $\dot{\alpha}_* = 2^{m+1}\dot{\alpha}/q_*^m$ as a function of α and m are shown in Fig. 1, and those of α as a function of $t_* = tq_*^m/2^{m+1}$ - in Fig. 2.

The dotted lines in the figures show the results according to the linear theory, and the solid lines those according to the nonlinear theory. As can be seen from the graphs, the discrepancies between the results of the two theories are too significant to be ignored. The derived variational equation (2.6) can be used not only for the solution of the problems of bending, but also for the problems of stability in the case of steady-state creep.

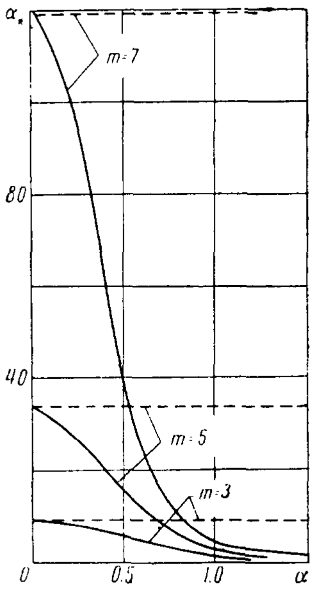

Fig. 1.

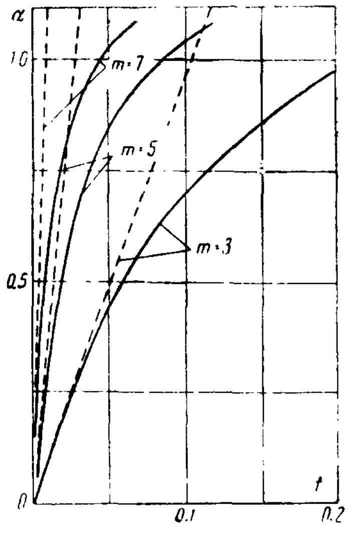

Fig. 2.

BIBLIOGRAPHY

1. Kachanov, L.M., *Teoriia polzuchesti (Theory of Creep)*. Fizmatgiz, 1960.
2. Hill, R., New horizons in the mechanics of solids. *J. Mechanics and Physics of Solids* 5, No. 1, 66-74, November, 1956.
3. Vol'mir, A.S., *Gibkie plastinki i obolochki (Flexible Plates and Shells)*. Gostekhizdat, 1956.

Translated by O.S.